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# PARTIAL DIFFERENTIAL EQUATIONS: FIRST-ORDER THEORY IN ONE DEPENDENT VARIABLE--DETERMINISTIC CASE

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## ABSTRACT

The basic theory of first-order partial differential equations in one dependent variable is developed from a differential geometric point of view in this report. This point of view is such that it holds promise for use in the stochastic case and in control theory.

## Introduction

Let M be a real $C^\infty$-manifold of dimension n and let F(M) be a manifold or bundle with projection $\pi$ onto M. We suppose that $\{\omega_i\}$ is a set of differential forms on F(M) and that V is a variety contained in F(M). The problem which we will examine (in a restricted manner), is: do there exist sections $\varphi$ of F(M) over M such that i) $\varphi^* (\omega_i)=0$ for all i and ii) $\varphi$ is a map into V? We shall attack this question in the case where F(M) is the first-order bundle over M and $\{\omega_i\}$ consists of a single form, $dz-q_i dy^i$.♀ This is, of course, the situation which represents the theory of first-order partial differential equations.

This paper consists of the following sections:

A.) Exact Sections and Solutions.

B.) The Basic Existence Theorem.

C.) Change of Variable.

D.) Poisson Brackets.

E.) Linear Systems.

F.) General Systems.

## A. Exact Sections and Solutions

Let M be a real $C^\infty$-manifold of dimension n and let F(M) be the first-order bundle over M with $\pi$ as the projection on M. Then: F(M) is of dimension 2n+1; the fiber of F(M) at p, $\tilde{F}(M)_p$, $p \in M$ is (isomorphic with) $R \oplus \tilde{D}_p$ where $\tilde{D}_p$ is the fiber of the dual bundle D of M at p and R is the reals. If $p \in M$ and U is a

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♀ The summation convention will be used throughout.

coordinate neighborhood of M at p with $x^1, \ldots, x^n$ as coordinate functions, then, on $\pi^{-1}(U)$, the coordinate functions are $z, q_1, \ldots, q_n, y^1, \ldots y^n$ where z is the R-coordinate, $y^i = x^i \circ \pi$, and $w_q \in \tilde{D}_q$ implies that $w_q = q_i(q) d x^i_q$.

Definition 1: Let $\omega = dz - q_i dy^i$ and let $\varphi$ be an element of $\Gamma(U, F(M))$ (i.e. $\varphi$ is a section of F(M)). Then $\varphi$ is exact if $\varphi^*(\omega) = 0$.

Definition 2: Let f be a smooth function on M such that the domain of f contains U. Then the section defined by f, $\varphi_f$, is the element of $\Gamma(U, F(M))$ given by:

$$\varphi_f(p) = (f(p), (D_1 f)(p), \ldots, (D_n f)(p), x^1(p), \ldots, x^n(p))$$

where p is an element of U and the $D_i$ are partial differentiations.

Proposition 1: A section $\varphi$ is exact if and only if there is a smooth function f such that $\varphi = \varphi_f$.

Proof: We first observe that if $\Psi$ is in $\Gamma(U, F(M))$ and p is an element of U, then $\Psi^*_p(\omega_{\Psi(p)}) = d(z \circ \Psi)_p - q_i(\Psi(p))\, d(y^i \circ \Psi)_p$ and, therefore, $\Psi^*_p(\omega_{\Psi(p)}) = 0$ if and only if $q_i(\Psi(p)) = D_i(z \circ \Psi)(p)$. It follows immediately that $\varphi_f$ is exact if f is smooth (since $q_i \circ \varphi_f = D_i f = D_i(z \circ \varphi_f)$).

On the other hand, if $\varphi$ is exact, then $z \circ \varphi = f$ is a smooth function and $\varphi = \varphi_f$ since $q_i \circ \varphi = D_i(z \circ \varphi) = D_i f$.

Proposition 2: $\omega = dz - q_i dy^i$ is an intrinsic form.

Proof: It will suffice to show that $q_i dy^i$ is an intrinsic form since dz is intrinsic. Let q be a fixed point of F(M) with $p = \pi(q)$ its projection on M. Then (at q and p) we have:

$$\bar{\pi}(Z)(x^j) = Z(x^j \circ \pi) = Zy^j = 0$$

$$\bar{\pi}(Q^i)(x^j) = Q^i(x^j \circ \pi) = Q^i y^j = 0$$

$$\bar{\pi}(Y_i)(x^j) = Y_i(x^j \circ \pi) = Y_i y^j = \delta^i_j$$

These completely describe the behavior of $\pi$ since $Z, Q^i, Y_i$ form a basis of $F(M)_q^{♀}$ and if f is a smooth function on M, the $f = F \circ (x)$ and $f \circ \pi = F \circ (x) \circ \pi = F \circ (y)$.

Now suppose that $q = (z_q, w_q)$ with $w_q \epsilon \tilde{D}_p$. We define the map $\Psi_q$ of $F(M)_q$ into R as follows: $\Psi_q(T_q) = w_q(\bar{\pi}(T_q))$, $T_q \epsilon F(M)_q$ and we set $\Psi(q) = \Psi_q$. Then $\Psi$ maps F(M) into the dual bundle to F(M) and is, in fact, a covector field. We claim it is $q_i dy^i$ which will establish the proposition since $\Psi$ is intrinsic.

However, we have (at q and p):

$$(q_i dy^i)(Z) = q_i Zy^i = 0$$

$$(q_i dy^i)(Q^j) = q_i Q^j y^i = 0$$

$$(q_i dy^i)(Y_j) = q_i Y_j y^i = q_j$$

$$\Psi(Z) = w(\bar{\pi}(Z)) = w(0) = 0$$

$$\Psi(Q^j) = w(\bar{\pi}(Q^j)) = w(0) = 0$$

$$\Psi(Y_j) = w(\bar{\pi}(Y_j)) = q_i(dx^i)(X_j) = q_j$$

and our claim is valid.

Definition 3: Let V be a subvariety of F(M) which is regular (i.e. $\bar{\pi}$ has rank equal to the dimension of V on V). Then a solution of the equations which define

♀ The tangent space to F(M) at q.

V is an exact section $\varphi$ whose range lies in V.

Proposition 3: If $p \in M$, if $\varphi$ is a section of $F(M)$ whose domain contains p, and if $\Omega_{\varphi(p)}$ is the subspace of $F(M)_{\varphi(p)}$ annihilated by $\omega_{\varphi(p)}$, then a necessary and sufficient condition that $\varphi$ be exact at p is: $\overline{\varphi}_p \ (M_p) \subset \Omega_{\varphi(p)}$.

Proof: If $T \in M_p$, then $\omega_{\varphi(p)} (\overline{\varphi}_p (T)) = [\, d(z \circ \varphi) - (q_i \circ \varphi)\, d(y^i \circ \varphi)\,]_p (T) = \varphi^*_p (\omega_{\varphi(p)}) (T)$. It follows that $\varphi^*_p \ (\omega_{\varphi(p)}) = 0$ if and only if $\omega_{\varphi(p)} (\overline{\varphi}_p (T)) = 0$ for all T in $M_p$, i.e., if and only if $\overline{\varphi}_p \ (M_p) \subset \Omega_{\varphi(p)}$.

B. The Basic Existence Theorem

We shall prove, in this section, the basic local existence theorem for a single equation in one dependent variable.

Definition 4: Let $\Gamma$ be a $C^\infty$ curve on F(M) with t as parameter and let $\Phi$ be a smooth function on F(M). Then $\Gamma$ is an integral curve of $\Phi$ if $\Phi \circ \Gamma = 0$ and $\Gamma$ is a characteristic curve of $\Phi$ if:

$$\overline{\Gamma} D_t{}^{♀} = (Q^i \Phi) Y_i - \{Y_i \Phi + q_i Z \Phi\} Q^i + q_i Q^i \Phi Z$$

or, equivalently, if:

$$(\overline{\Gamma} D_t)(y^i) = Q^i \Phi$$
$$(\overline{\Gamma} D_t)(q_i) = - Y_i \Phi - q_i Z \Phi$$
$$(\overline{\Gamma} D_t)(z) = q_i Q^i \Phi$$

We observe that if $\Gamma$ is a characteristic curve of $\Phi$, then $\overline{\Gamma} D_t$ is a smooth vector field on F(M) which is not intrinsic.

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♀ $D_t$ is the canonical vector field on $\Gamma$ obtained from differentiation with respect to t.

<u>Proposition 4:</u> If $\Gamma$ is a characteristic curve of $\Phi$, then (a) $\omega(\bar{\Gamma}D_t) = 0$ and (b) if $q_o = \Gamma(t_o)$ and $\Phi(q_o) = 0$, then $\Gamma$ is also an integral curve of $\Phi$.

<u>Proof:</u> (a) We see, from the definitions, that:

$$\omega(\bar{\Gamma}D_t) = (dz - q_i dy^i)(Q^j \Phi Y_j - \{Y_j \Phi + q_j Z\Phi\} Q^j + q_i Q^j \Phi Z)$$

$$= Q^j \Phi Y_j z - Y_j \Phi Q^j z - q_j Z\Phi Q^j z + q_j Q^j \Phi Z z -$$

$$q_i Q^j \Phi Y_j y^i + q_i Y_j \Phi Q^j y^i + q_i q_j Z\Phi Q^j y^i - q_i q_j Q^j \Phi Z y^i$$

$$= q_j Q^j \Phi - q_i Q^j \Phi \delta_j^i = 0$$

(b) We observe that $D_t(\Phi \circ \Gamma) = (\bar{\Gamma}D_t)(\Phi) = Q^i \Phi Y_i \Phi - Y_i \Phi Q^i \Phi - q_i Z\Phi Q^i \Phi + q_i Q^i \Phi Z \Phi = 0$. It follows that $\Phi$ is constant along $\Gamma$ and our assertion has been demonstrated.

Let U be a coordinate region in M with $x^1, \ldots, x^n$ as coordinate functions and let N be an n-1 dimensional submanifold of M such that $N \cap U$ is given by the equation $x^n = H(x^1, \ldots, x^{n-1})$. Let g be a smooth function on M whose domain contains U and suppose that $g = h(x^1, \ldots, x^{n-1})$ on $N \cap U$. If $\Phi$ is a smooth function on F(M) whose domain contains $\pi^{-1}(U)$, then we wish to find conditions which insure the existence of a section $\varphi$ of F(M) over some $U' \subset U$ such that $\Phi \circ \varphi = 0$ and $\varphi = \varphi_g$ on $N \cap U$ (i. e., we wish to solve the local initial value problem for the equation $\Phi = 0$).

<u>Definition 5:</u> A <u>special neighborhood</u> of $\{0\} \times N \cap U$ in $R \times N \cap U$ is an open neighborhood W of $\{0\} \times N \cap U$ for which $R \times \{p\} \cap W$ is connected for all p in $N \cap U$.

We observe that if $\Sigma |Q^i \Phi| \neq 0$ on $\pi^{-1}(U)$, then there is a special neighborhood

W of $\{0\} \times N \cap U$ and there is a smooth map $\gamma$ of $\mathbf{W}$ into F(M) such that : (a) $\gamma(0, p) = \varphi_g(p)$ for p in $N \cap U$; and (b) if $\Gamma_p(t) = \gamma(t, p)$ for p in $N \cap U$, then $\Gamma_p$ is a characteristic curve of $\Phi$.♀

Theorem 1: If $\Sigma |Q^i \Phi| \neq 0$ on $\pi^{-1}(U)$, if $\Phi \circ \varphi_g = 0$ on $N \cap U$, if $\mathbf{W}$ is a special neighborhood of $\{0\} \times N \cap U$ and $\gamma$ is a smooth map of $\mathbf{W}$ into F(M) such that $\gamma(0, p) = \varphi_g(p)$, for p in $N \cap U$, and $\Gamma_p(t) = \gamma(t, p)$ $(p \in N \cap U)$ is a characteristic curve of $\Phi$, if $\nu$ is the map of $\mathbf{W}$ into U given by $\nu(t, p) = \pi(\gamma(t, p))$, if $\mathbf{W}' \subset \mathbf{W}$ is a neighborhood of $\{0\} \times N \cap U$ on which $\nu$ is one-one,♀♀ and if $U' = \nu(\mathbf{W}')$, then there is a smooth function f on M such that $\Phi \circ \varphi_f = 0$ on U' and $\varphi_f = \varphi_g$ on $N \cap U$.

Proof: Let $\Psi$ be the map of U' into F(M) given by: $\Psi(p) = (\gamma \circ \nu^{-1})(p)$. Then we assert that $\Psi$ has the following properties:

(a) $\Psi$ is in $\Gamma(U', F(M))$ i.e., $\Psi$ is a section

(b) $\Phi \circ \Psi = 0$

(c) $\Psi = \varphi_g$ on $N \cap U$

(d) $\Psi$ is exact.

Assuming that this assertion has been established, we see that $f = z \circ \Psi$ satisfies the conclusions of the theorem. We now establish the validity of our assertion.

(a) If $q \in U'$ with $q = \pi(\gamma(t, p))$, then $\Psi(q) = (\gamma \circ \nu^{-1})(q) = \gamma(t, p)$. It follows that $\pi(\Psi(q)) = \pi(\gamma(t, p)) = q$ and hence, that $\Psi$ is a section (since $\Psi$ is clearly smooth).

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♀ See Coddington and Levinson [ 1] p. 25

♀♀ $\mathbf{W}'$ exists since $\nu$ is smooth, $\nu$ restricted to $\{0\} \times \mathbf{W} \cap U$ is one-one and $\Sigma |Q^i \Phi| \neq 0$ on $\pi^{-1}(U)$.

(b) If $q \in U'$ with $q = \pi(\gamma(t, p))$, then $\Psi(q) = \gamma(t, p) = \Gamma_p(t)$ and therefore, $(\Phi \circ \Psi)(q) = \Phi(\Gamma_p(t)) = (\Phi \circ \Gamma_p)(t)$. But, $(\Phi \circ \Gamma_p)(0) = \Phi(\gamma(0, p)) = \Phi(\varphi_g(p)) = 0$ and $\Gamma_p$ is a characteristic curve of $\Phi$; it follows (Proposition 4) that $(\Phi \circ \Gamma_p)(t) = 0$ and consequently, that $\Phi \circ \Psi = 0$.

(c) If $p \in N \cap U$, then $p = \pi(\gamma(0, p)) = \nu(0, p)$ and so we have $\Psi(p) = \gamma(0, p) = \varphi_g(p)$.

(d) Let $q \in U'$ with $q = \pi(\gamma(t, p))$. We observe, first of all, that $\bar{\Psi}_q(M_q) = \bar{\gamma}_{(t,p)}[\bar{\nu}^{-1}(M_q)] = \bar{\gamma}_{(t,p)}(W'_{(t,p)})$. In view of Proposition 3, it will suffice to show that $\bar{\Psi}_q(M_q) = \bar{\gamma}_{(t,p)}(W'_{(t,p)}) \subset \Omega_{\Psi(q)} = \Omega_{\gamma(t,p)}$. We note now that $X_{1p}, \ldots, X_{n-1p}, D_t$ are a basis of $W'_{(t,p)}$ and therefore, we need only demonstrate that:

$$X_{i(t,p)}(z \circ \gamma) - (q_j \circ \gamma) X_{i(t,p)}(y^j \circ \gamma) = \omega_{\gamma(t,p)}(X_i) = 0, \quad i = 1, \ldots, n-1$$

and that:

$$D_t(z \circ \gamma) - (q_j \circ \gamma) D_t(y^j \circ \gamma) = \omega_{\gamma(t,p)}(\bar{\Gamma}_p D_t) = 0.$$

Let p be fixed and define the functions $\xi_i(t)$ as follows:

$$\xi_i(t) = X_{i(t,p)}(z \circ \gamma) - (q_j \circ \gamma) X_{i(t,p)}(Y^j \circ \gamma).$$

then we observe that $\xi_i(0) = X_{i(0,p)}(z \circ \gamma) - (q_j \circ \gamma)(0, p)(X_{i(0,p)}(y^j \circ \gamma) = X_{ip}\, g - X_{jp}\, g\, \delta_i^j = 0$.

Now, we also see that:

$$\begin{aligned} D_t\, \xi_i(t) &= D_t X_i (z \circ \gamma) - D_t[(q_j \circ \gamma) X_i(y^j \circ \gamma)] \\ &= X_i D_t (z \circ \gamma) - D_t(q_j \circ \gamma) X_i(y^j \circ \gamma) - (q_j \circ \gamma) X_i D_t (y^j \circ \gamma) \\ &= X_i [(q_j \circ \gamma) D_t (y^j \circ \gamma)] - D_t(q_j \circ \gamma) X_i(y^j \circ \gamma) - (q_j \circ \gamma) X_i D_t(y^j \circ \gamma)^{♀} \end{aligned}$$

♀ Since $\gamma(t, p) = \Gamma_p(t)$ is a characteristic curve of $\Phi$.

$$= X_i(q_j \circ \gamma) Q^j \Phi - D_t(q_j \circ \gamma) X_i(y^j \circ \gamma)$$

$$= X_i(q_j \circ \gamma) Q^j \Phi - \{Y_j \Phi + (q_j \circ \gamma) Z\Phi\} X_i(y^j \circ \gamma).$$

However, $\Phi \circ \gamma = 0$ implies that $Y_j \Phi X_i(y^j \circ \gamma) + Z\Phi X_i(z \circ \gamma) + Q^j \Phi X_i(q_j \circ \gamma) = 0$ and therefore, that $D_t \xi_i(t) = -(Z\Phi)\xi_i(t)$. It follows that:

$$\xi_i(t) = \xi_i(0) \exp\left\{-\int_0^t (Z\Phi) \circ \Gamma_p(u)\, du\right\} = 0.$$

The fact that $\omega_{\gamma(t,p)}(\bar{\Gamma}_p D_t) = 0$ is an immediate consequence of Proposition 4. Thus, our result has been established.

C. Change of Variable

We shall show, in this section, that it suffices to consider equations (or systems) which are independent of z. More precisely, we shall show that if $\{\Phi_i\}$ is a set of smooth functions on F(M) and the equations $\Phi_i = 0$ define the variety V, then the assumption $Z\Phi_i = 0$ for all i causes no loss of generality. Our procedure corresponds to the classical change of variable procedure and will serve to simplify the sequel.

Proposition 5: If $\Phi = 0$ is an equation on F(M) for which $Z\Phi = 0$, then there is an equation $\Phi_1 = 0$ on D(M)♀ such that a smooth function f on M is a solution of $\Phi = 0$ if and only if f is a solution of $\Phi_1 = 0$.

Proof: We observe that, in view of the relation $Z\Phi = 0$, there exists a $C^\infty$ function $\Phi^*$ on $R^{2n}$ such that $\Phi_1 = \Phi^* \circ (q, y)$ may be viewed as a smooth function on D(M) and we consider the equation $\Phi_1 = 0$ on D(M).

Suppose that f is a solution of $\Phi = 0$. Then:

$$
\begin{aligned}
0 &= \Phi(\varphi_f(p)) \quad \text{(for p in the domain of f)} \\
&= \Phi(f(p), (D_1 f)(p), \ldots, (D_n f)(p), x^1(p), \ldots, x^n(p)) \\
&= \Phi^*((D_1 f)(p), \ldots, D_n f)(p), x^1(p), \ldots, x^n(p)) \\
&= \Phi^*(\psi_f(p))
\end{aligned}
$$

♀ The dual bundle over M.

where $\Psi_f$ is the section of D(M) given by $\Psi_f(p)=((D_1f)(p),\ldots,(D_nf)(p), x^1(p),\ldots x^n(p))$.

We further note that if $\pi_1$ is the projection of F(M) on D(M), then $\Phi_1 \circ \pi_1 = \Phi$. It follows immediately that a solution of $\Phi_1=0$ is again a solution of $\Phi=0$.

We shall now show how to take an equation $\Phi=0$ on F(M), which may depend on z, and "extend" it to an equation (or system of equations) on the dual bundle $D(M_1)$ of some $C^\infty$-manifold $M_1$ (having M as a submanifold) preserving solutions in the process. We let $M_1=M\times R$. Then we assert that there is a pair of equations on $F(M_1)$, both independent of the R-coordinate on $F(M_1)$, such that each solution of $\Phi=0$ defines a solution of this pair of equations and conversely. Assuming that we have demonstrated the validity of this assertion, it follows, by virtue of Proposition 5, that it is sufficient to consider equations in D(M).

First of all, we note that $D(M_1)$ can be identified, in a natural way, with $F(M)\times R$. If $\langle p,t\rangle \in M_1$, then the fiber of $F(M)\times R$ over $\langle p,t\rangle$ is $\tilde{F}(M)_p\times\{t\}$ which can be naturally identified with $\tilde{D}(M)_p \oplus R\times\{t\}$. We set $\Psi\langle p,t\rangle((w_p,s))=(w_p,s,t)$ where $(w_p,s)$ is an element of $\tilde{D}(M_1)\langle p,t\rangle$, and we let $\Psi$ be the union of the maps $\Psi\langle p,t\rangle$. Then it is clear that $\Psi$ is a one-one map of $D(M_1)$ <u>onto</u> $F(M)\times R$. Furthermore, $\Psi$ is an invariant map and both $\Psi$ and its inverse preserve smooth functions. It follows that $D(M_1)$ may be identified with $F(M)\times R$ and hence, that $F(M_1)$ and $F(M)\times R\times R$ may also be identified.

It is clear that the map $\delta$ of $F(M)\times R$ into $F(M)\times R$ is given by $\delta(w_p,s,t)=(w_p,s,t)$ is a bundle isomorphism. We set $\beta=\delta\circ\Psi$ and we note that $\beta$ is also a bundle isomorphism. If **A** is a manifold and if B is a bundle over **A**, then we denote the projection of B on **A** by $\pi(B,A)$. We observe that the following diagram is commutative.

$$
\begin{tikzcd}
F(M_1) \arrow[r, "\pi(F(M_1), D(M_1))"] \arrow[dr, "\pi(F(M_1), M_1)"'] & D(M_1) \arrow[r, leftrightarrow, "\beta"] \arrow[d, "\pi(D(M_1), M_1)"] & F(M)\times R \arrow[r, "\pi(F(M)\times R, F(M))"] & F(M) \arrow[d, "\pi(F(M), M)"] \\
 & M_1 \arrow[rr, "\pi(M_1, M)"] & & M
\end{tikzcd}
$$

Suppose that f is a smooth function on M, then the mapping $\sigma_f$ of M into $M_1$ given by $\sigma_f(p) = (p, f(p))$, $p \in M$, is a section of $M_1$ over M which maps M onto a submanifold of $M_1$. We observe that $f \circ \pi(M_1, M)$♀, when restricted to $\sigma_f(M)$, is a smooth function which can be extended to a smooth function on $M_1$. We denote this function by $f \circ \sigma_f^{-1}$. On the other hand, suppose that g is a smooth function on $M_1$ which is independent of the R-coordinate on $M_1$, then the mapping $\tau_g$ of M into $M_1$ given by $\tau_g(p) = (p, g(p, 0))$, $p \in M$, is a section of $M_1$ over M which maps M onto a submanifold of $M_1$. We observe that $g \circ \tau_g$ is a smooth function on M.

Theorem 2: Let $\Phi$ be a smooth function on F(M) and let $\Phi_1 = \Phi \circ \pi(F(M) \times R, F(M)) \circ \beta \circ \pi(F(M_1), D(M_1))$. If the equation $\Phi=0$ has f as a solution, then $f \circ \sigma_f^{-1}$ is a solution of the equations $\Phi_1=0$ and $q_{n+1}=0$ on $F(M_1)$.♀♀ Conversely, if g is a solution of the equations $\Phi_1=0$ and $q_{n+1}=0$, then $g \circ \tau_g$ is a solution of the equation $\Phi=0$. Furthermore, the mapping $f \to f \circ \sigma_f^{-1}$ is a one-one mapping of the set of solutions of the equation $\Phi=0$ onto the set of solutions of the system of equations $\{\Phi_1=0, q_{n+1}=0\}$.

Proof: We observe that $f \circ \sigma_f^{-1}$ is independent of the R-coordinate on $M_1$ and hence, that $D_{n+1} f \circ \sigma_f^{-1} = 0$ (i.e. $f \circ \sigma_f^{-1}$ is a solution of $q_{n+1}=0$). We also note that $\varphi_{f \circ \sigma_f^{-1}}(p, t) = \varphi_{f \circ \sigma_f^{-1}}(p, f(p))$, $p \in M$, and so we have:

$$\begin{aligned}\Phi_1(\varphi_{f \circ \sigma_f^{-1}}(p, f(p))) &= \Phi_1(f(p), (D_1 f)(p), \ldots, (D_n f)(p), 0, x^1(p), \ldots, x^n(p), f(p)) \\ &= \Phi(f(p), (D_1 f)(p), \ldots, (D_n f)(p), x^1(p), \ldots, x^n(p)) \\ &= \Phi(\varphi_f(p)) \\ &= 0.\end{aligned}$$

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♀ $= f \circ \sigma_f^{-1}$

♀♀ or, equivalently, in view of Propositions 5, of the equations $\Phi \circ \pi(F(M) \times R, F(M)) \circ \beta = 0$ and $q_{n+1}=0$ on $D(M_1)$.

The first point has, therefore, been established.

On the other hand, if g is a solution of $\Phi_1=0$ and $q_{n+1}=0$, then $\varphi_{g \circ \tau_g}(p) = (g(p, g(p, 0)), (D_1 g)(p, g(p, 0)), \ldots, (D_n g)(p, g(p, 0)), x^1(p), \ldots x^n(p))$ and, consequently, $\Phi(\varphi_{g \circ \tau_g}(p)) = \Phi_1(g(p, g(p, 0)), (D_1 g)(p, g(p, 0)), \ldots, (D_n g)(p, g(p, 0)), 0, x^1(p), \ldots, x^n(p), g(p, 0))$ ♀ $= \Phi_1(\varphi_g(p, g(p, 0))) = 0$. Thus, we have established the second point.

Finally, we note that the map $f \to f \circ \sigma_f^{-1}$ is clearly one-one and that $g \circ \tau_g \circ \sigma_{g \circ \tau_g}^{-1}(p, t) = g \circ \tau_g(p) = g(p, g(p, 0)) = g(p, t)$ since g is independent of the R-coordinate on $M_1$. This completes the proof of the theorem.

D. Poisson Brackets

We shall introduce, in this section, the Poisson bracket operation. Let $\Phi$ and $\Psi$ be smooth functions on F(M) which are independent of the R-coordinate on F(M) and let U be a coordinate region on M with $x^1, \ldots, x^n$ as coordinate functions on U and with $z, q_1, \ldots, q_n, y^1, \ldots, y^n$ as coordinate functions on $\pi^{-1}(U)$.

Definition 6: The Poisson bracket of $\Phi$ and $\Psi$ (in symbols: $(\Phi, \Psi)$) on $\pi^{-1}(U)$ is the function $Y_j \Phi Q^j \Psi - Y_j \Psi Q^j \Phi$.

We observe that the intrinsic differential form $d\omega$ $(= - dq_j \wedge dy^j$ on $\pi^{-1}(U))$ on F(M) (or D(M)) induces a non-singular bilinear form on $D(M)_w \oplus D(M)_w$. ♀♀ It follows that $d\omega$ defines an isomorphism $\nu_w$ between $D(M)_w$ and its dual $D(M)_w{}^*$. We let $\nu$ be the map generated by the $\nu_w$. We then note that, if $\Theta$ is a smooth function on F(M) which is independent of the R-coordinate on F(M), then $d\,\Theta_w$ can be viewed as an element of $D(M)_w{}^*$ and consequently, $\nu_w^{-1}(d\Theta_w)$ is a well-defined element of

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♀ Noting that g is independent of the R-coordinate on $M_1$.

♀♀ Where $D(M)_w$ is the tangent space to D(M) at w.

$D(M)_w$.

Proposition 6: The bracket operation is intrinsic and well-defined. In fact, $(\Phi, \Psi) = d\omega\,(\nu^{-1}(d\Phi), \nu^{-1}(d\Psi))$.

Proof: We fix $w_0$ in D(M) and we shall omit the subscript $w_0$ in the rest of the proof. First of all, we note that $\nu(Y_i) = dq_i$ and $\nu(Q^i) = -\, dy^i$ since:

$$
\begin{aligned}
d\omega\,(Y_i, Y_k) &= dy^j(Y_i\, dq_j\,(Y_k) - dy^j\,(Y_k) dq_j\,(Y_i) \\
&= (Y_i y^j)\,(Y_k q_j) - (Y_k y^j)\,(Y_i\, q_j) \\
&= 0 \\
d\omega\,(Y_i, Q^k) &= dy^j\,(Y_i)\, dq_j (Q^k) - dy^j\,(Q^k)\, dq_j\,(Y_i) \\
&= (Y_i y^j)\,(Q^k q_j) - (Q^k y^j)\,(Y_i q_j) \\
&= \delta^k_i \\
d\omega\,(Q^i, Q^k) &= dy^j\,(Q^i)\, dq_j\,(Q^k) - dy^j\,(Q^k)\, dq_j\,(Q^i) \\
&= (Q^i y^j)\,(Q^k q_j) - (Q^k y_j)\,(Q^i q_j) \\
&= 0 \\
d\omega\,(Q^i, Y_k) &= dy^j\,(Q_i)\, dq_i\,(Y_k) - dy^j\,(Y_k)\, dq_j\,(Q^i) \\
&= (Q^i y^j)\,(Y_k q_j) - (Y_k y^j)\,(Q^i q_j) \\
&= -\,\delta^i_k .
\end{aligned}
$$

Now, we know that $d\,\Phi = Q^i\,\Phi\, dq_i + Y_i\,\Phi\, dy^i$ and $d\Psi = Q^i\,\Psi\, dq_i + Y_i\,\Psi\, dy^i$. It follows that $\nu^{-1}(d\Phi) = Q^i\Phi\, Y_i - Y_i\,\Phi\, Q^i$ and that $\nu^{-1}\,(d\Psi) = Q^i\,\Psi\, Y_i - Y_i\,\Psi\, Q^i$. Therefore, we have:

$$d\omega(\nu^{-1}(d\Phi), \nu^{-1}(d\Psi)) = d\omega\,(Q^i\Phi Y_i - Y_i\Phi Q^i, Q^j\Psi Y_j - Y_j\Psi Q^j)$$

$$= Q^i\Phi Q^j\Psi d\omega\,(Y_i, Y_j) - Y_i\,\Phi Q^j\Psi\; d\omega\,(Q^i Y_j)$$

$$= Q^i\,\Phi Y_j\Psi\; d\omega\,(Y_i, Q^j) + Y_i\Phi\; Y_j\Psi\; d\omega\,(Q^i, Q^j)$$

$$= Y_i\Phi Q^j\Psi\,\delta_j^{\ i} - Y_j\Psi Q^i\Phi\delta_j^{\ i}$$

$$= Y_j\Phi Q^j\Psi - Y_j\Psi Q^j\Phi$$

$$= (\Phi, \Psi).$$

Proposition 7: If $\Phi_1, \Phi_2, \Phi_3$ are smooth functions on F(M) which are all indpendent of the R-coordinate on F(M), then:

a) $(\Phi_1+\Phi_2, \Phi_3) = (\Phi_1, \Phi_3) + (\Phi_2, \Phi_3)$

b) $(\Phi_1\Phi_2, \Phi_3) = (\Phi_1, \Phi_3)\,\Phi_2 + (\Phi_2, \Phi_3)\,\Phi_1$

c) $(\Phi_1, \Phi_2) = -(\Phi_2, \Phi_1)$

d) $(\Phi_1, (\Phi_2, \Phi_3)) + (\Phi_2, (\Phi_3, \Phi_1)) + (\Phi_3, (\Phi_1, \Phi_2)) \equiv 0$

(the Jacobi identity).

Proof: The properties a), b), and c) are an immediate consequence of the definition of the bracket operation.

To prove d), we first note that $(\Phi_1, (\Phi_2, \Phi_3)) + (\Phi_2, (\Phi_3, \Phi_1)) + (\Phi_3, (\Phi_1, \Phi_2)) =$

$$\frac{(\Phi_1, (\Phi_2, \Phi_3)) + (\Phi_2, (\Phi_3, \Phi_1))}{2} + \frac{(\Phi_2, (\Phi_3, \Phi_1)) + (\Phi_3, (\Phi_1, \Phi_2))}{2} + \frac{(\Phi_1, (\Phi_2, \Phi_3)) + (\Phi_3, (\Phi_1, \Phi_2))}{2}.$$

We next observe that expressions of the form $(\Phi_i, (\Phi_j, \Phi_k)) - (\Phi_j, (\Phi_i, \Phi_k))$ $(= (\Phi_i, (\Phi_j, \Phi_k)) + (\Phi_j, (\Phi_k, \Phi_i)))$ contain no terms involving "multiple derivatives" (e.g., of the form $Y_j\Psi^k(\cdot)$). However, expressions of the form $(\Phi_i, (\Phi_j, \Phi_k))$ do contain "multiple derivatives". The assertion d) follows immediately.

## E. Linear Systems

We shall analyze linear systems primarily as an aid to the intuitive understanding of the more general results which will be developed in the next section. The most general linear system is defined by equations of the form $\Phi(z, q_1, \ldots, q_n, y^1, \ldots, y^n) = g^j(y^1, \ldots, y^n)\, q_j + g^0(y^1, \ldots, y^n) z - g(y^1, \ldots, y^n) = 0$; we observe, however, that we may suppose that $Z\Phi = 0$ (i.e., that $g^0(y^1, \ldots, y^n) \equiv 0$) in view of section C.

Definition 7: A linear system defined by a set of equations of the form.♀

$$(*) \quad \Phi_k(z, q_1, \ldots, q_n, y^1, \ldots, y^n) = g_k^j(y^1, \ldots, y^n)\, q_j = 0,$$

is called a homogeneous linear system.

We will show, in an appendix to this section, that it is sufficient to consider homogeneous linear systems and as a consequence, we shall consider only such systems in the remainder of this section.

Definition 8: A homogeneous linear system is independent if $\varphi^k \Phi_k \equiv 0$ implies that $\varphi^k \equiv 0$ where the $\varphi^k$ are smooth functions on F(M) which depend only on $y^1, \ldots, y^n$.

We remark that the set of homogeneous linear functions on F(M) may be viewed as a vector space (module) over the ring of smooth functions on M which gives us a natural notion of independence. We further observe that any homogeneous linear function on F(M) depends upon $q_1, \ldots, q_n$ and hence, the module of homogeneous linear functions is of dimension n. If $\Psi_\alpha$ is a set of homogeneous linear functions on F(M), then we denote by, sp $(\Psi_\alpha)$, the span of the $\Psi_\alpha$. We see immediately that any solution of the system $\{\Psi_\alpha = 0\}$ is also a solution of any equation $\Psi = 0$ with $\Psi$ in sp $(\Psi_\alpha)$. It follows that we need only consider homogeneous linear systems (*)

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♀ on $\pi^{-1}(u)$ where u is a coordinate region on M.

in which the $\Phi_k$ are independent. Finally, we observe that the Poisson bracket of two homogeneous linear functions is again a homogeneous linear function.

Definition 9: A homogeneous linear system(*) is complete if it is independent and if $(\Phi_i, \Phi_j)$ is an element of $sp(\Phi_k)$ for all i, j.

Proposition 8: If $\Phi_k = 0$ is a homogeneous linear system which is complete and if T is a non-singular linear transformation which leaves $sp(\Phi_k)$ invariant, then $T(\Phi_k) = 0$ is a complete homogeneous linear system.

Proof: Let $T(\Phi_k) = \varphi_k^j \Phi_j$ and let $\| \eta_s^r \|$ be the inverse of the matrix $\| \varphi_k^j \|$. Then we have:

$$\begin{aligned}(T(\Phi_i), T(\Phi_k)) &= (\varphi_i^\nu \Phi_\nu, \varphi_k^\mu \Phi_\mu) \\ &= \varphi_i^\nu \varphi_k^\mu (\Phi_\nu, \Phi_\mu) + \varphi_i^\nu \Phi_\mu (\Phi_\nu, \varphi_k^\mu) \\ &\quad + \varphi_k^\mu \Phi_\nu (\varphi_i^\nu, \Phi_\mu) + \Phi_\nu \Phi_\mu (\varphi_i^\nu, \varphi_k^\mu)\end{aligned}$$

However, we note that $\Phi_\mu = \eta_\mu^r T(\Phi_r)$ and that $\Phi_\nu = \eta_\nu^r T(\Phi_r)$. It follows that if $(\Phi_\nu, \Phi_\mu) = \alpha_{\nu\mu}^t \Phi_t$, then:

$$\begin{aligned}T(\Phi_i), T(\Phi_k)) = \varphi_i^\nu \varphi_k^\mu \alpha_{\nu\mu}^t \eta_t^r T(\Phi_r) + \varphi_i^\nu (\Phi_\nu, \varphi_k^\mu) \eta_\mu^r T(\Phi_r) \\ + \varphi_k^\mu (\varphi_i^\nu, \Phi_\mu) \eta_\nu^r T(\Phi_r) .\end{aligned}$$

We next observe that $(\Phi_\nu, \varphi_k^\mu)$ and $(\varphi_i^\nu, \Phi_\mu)$ are actually smooth functions on M from which the proposition follows.

Proposition 9: If $\Phi_k = 0$ is a homogeneous linear system which is complete and if $y^i = h^i(v^i, \dots, v^n)$ is a smooth change of variable with non-vanishing Jacobian, then the transformed system is complete.

Proof: We let $r_i$ be the coordinates relative to the $v^i$ which correspond to the $q_i$.

Then we see that $q_\alpha = (Y_\alpha v^i) r_i$ and hence, that the transformed system is:

$(g^\alpha_k Y_\alpha v^i) r_i = 0$. It follows that $\Phi_k = (g^\alpha_k Y_\alpha v^i) r_i$ and therefore, that independence is preserved. Moreover, $(\Phi_k, \Phi_\ell)$ is intrinsic. Our proposition follows immediately.

Definition 10: A homogeneous linear system (*) is involutory if it is independent and if $(\Phi_i, \Phi_j) \equiv 0$ for all i, j.

If $\Phi$ is a homogeneous linear function and if f is a smooth function on M, then we shall denote $\Phi(\varphi_f)$ by $A^\Phi(f)$.

Proposition 10: If $\Phi_k = 0$ is a homogeneous linear system and if $A^{\Phi_k}(f) = 0$ for all k, then $A^{\Phi_i}(A^{\Phi_j}(f)) - A^{\Phi_i}(A^{\Phi_j}(f)) = 0$.

Proof: Obvious.

We remark that $A^{\Phi_i}(A^{\Phi_j}(f)) - A^{\Phi_i}(A^{\Phi_j}(f)) = 0$ if and only if f is a solution of the equation $\{A^{\Phi_j}(g^\mu_i) - A^{\Phi_i}(g^\mu_j)\} q_\mu = 0$, i.e., of the equation $(\Phi_i, \Phi_j) = 0$.♀ It follows that the variety V defined by the system (*) may be replaced by the variety V' defined by the system $\Phi_k = 0$, $(\Phi_i, \Phi_j) = 0$ for all i, j. We next observe that $sp(\Phi_k) \subset$ $\subset sp(\Phi_k, (\Phi_i, \Phi_j)) \subset sp(\Phi_k, (\Phi_i, \Phi_j), (\Phi_k, (\Phi_i, \Phi_j)), ((\Phi_i, \Phi_j), (\Phi_r, \Phi_s))) \subset \ldots$ and hence, since the space of homogeneous linear functions has finite dimension n, this chain must terminate in at most n steps.

Definition 11: The completion of a homogeneous linear system (*) is "the" homogeneous linear system which corresponds to the terminal step in the chain $sp(\Phi_k) \subset$ $\subset sp(\Phi_k, (\Phi_i, \Phi_j)) \subset \ldots$ .

---

♀ Since $(\Phi_i, \Phi_j) = Y_\nu (g^\mu_i q_\mu) Q^\nu (g^\tau_j q_\tau) - Y_\nu (g^\tau_j q_\tau) Q^\nu (g^\mu_i q_\mu) = (D_\nu g^\mu_i) q_\mu g^\tau_j \delta^\nu_\tau - (D_\nu g^\tau_j) q_\tau g^\mu_i \delta^\nu_\mu = \{A^{\Phi_j}(g^\mu_i) q_\mu - A^{\Phi_i}(g^\tau_j) q_\tau\} = \{A^{\Phi_j}(g^\mu_i) - A^{\Phi_i}(g^\mu_j)\} q_\mu$.

We note that the completion of a homogeneous linear system (*) is complete and furthermore, a smooth function f on M is a solution of (*) if and only if it is a solution of the completion of (*). We may therefore suppose that the homogeneous linear systems, with which we are dealing, are complete.

Propostion 11: If $\Phi_k = 0$ is a complete homogeneous linear system, if $\| g_k^j \|$ has rank m and if (say) det $\| g_s^r \| \neq 0$ for $r, s = 1, \ldots, m$, then there is a linear transformation T, which is non-singular and which leaves $sp(\Phi_k)$ invariant, such that a) $T(\Phi_k) = \Psi_k = q_k + g_k^i q_j$, $k=1, \ldots, m$, $j=m+1, \ldots, n$ and b) $\Psi_k = 0$ is complete. Furthermore, $\Psi_k = 0$ is involutory if and only if $A^{\Psi_i}(g_k^i) - A^{\Psi_k}(g_i^j) \equiv 0$.

Proof: Assertion a) of the first part of the proposition is a well-known result of linear algebra and assertion b) is an immediate consequence of Proposition 8.

Since $(\Psi_k, \Psi_i) \equiv \{A^{\Psi_i}(g_k^i) - A^{\Psi_k}(g_i^j)\} q_j$, it follows that $\Psi_k = 0$ is involutory if and only if $\{A^{\Psi_i}(g_k^j) - A^{\Psi_k}(g_i^j)\} \equiv 0$ (since the $q_j$ are independent).

Finally, we observe that the completeness of the system $\Psi_k = 0$ is sufficient to insure the identical vanishing of $A^{\Psi_i}(g_k^j) - A^{\Psi_k}(g_i^j)$ since $(\Psi_k, \Psi_i) \equiv \varphi_{ki}^{\alpha} \Psi_{\alpha} \equiv \varphi_{ki}^{\alpha} q_{\alpha} + \varphi_{ki}^{\alpha} g_{\alpha}^j q_j \equiv \{A^{\Psi_i}(g_k^i) - A^{\Psi_k}(g_i^j)\} q_j$ and the $q_{\nu}$ are independent (i.e., $\varphi_{ki}^{\alpha}$ must be identically zero).

Definition 12: A homogeneous linear system of the form $\Psi_k = q_k + g_k^j q_j = 0$, $k=1, \ldots, m$, $j=m+1, \ldots, n$ which is complete (a fortiori involutory) is called canonical.

We observe that the substance of what has been done so far in this section is the reduction of the problem of the existence of solutions of a general linear system to the problem of the existence of solutions of a canonical linear system. Our next theorem pertains to the reduced problem.

Let U be a coordinate region in M with $x^1, \ldots, x^n$ as coordinate functions and let N be an n-m dimensional submanifold of M such that $N \cap U$ is given by the equation $x^k = c^k$, $k=1, \ldots, m$. Let h be a smooth function on M whose domain contains U and suppose that $h = h(x^{m+1}, \ldots, x^n)$ on $N \cap U$. If $\Psi_k$, $k=1, \ldots, m$ is a family of smooth (linear homogeneous) functions on F(M) whose domains contain $\pi^{-1}(U)$ and if the system $\Psi_k = 0$ is canonical, then we wish to find conditions which insure the existence of an (exact) section $\varphi$ of F(M) over some $U' \subset U$ such that $\Psi_k \circ \varphi = 0$ and $\varphi = \varphi_n$ on $N \cap U$ (i.e., we wish to solve the local initial value problem for the system of equations $\Psi_k = 0$).

<u>Theorem 3:</u> If $\Psi_k = 0$ is a canonical system, then there is an open $U' \subset U$ and a smooth function f on M such that f is a solution of the local initial value problem for the system $\Psi_k = 0$ on U'.

<u>Proof:</u> We let I denote the interval [0, 1] and we consider the mapping $\sigma$ of M×I into M given by: $\sigma((v^k(q), x^{m+j}(q), v(q))) = (c^k + v(q)v^k(q), x^{m+j}(q))$. Then $\sigma$ is a smooth mapping. We shall consider the equation $\Phi = r + v^k g_k^j q_j = 0$ on M×I, where r is the coordinate which corresponds to dv on I, with the initial condition $H(v^k, x^{m+j}, 0) = h(x^{m+1}, \ldots, x^n)$.

We observe that the conditions of Theorem 1 hold for the equation $\Phi=0$ and that the solution F of this equation is defined on an open set of the form $U^* \times I$ (since $\Phi$ is linear). We next remark that the restriction f of F to $M\times\{1\}$ is a solution of the system $\Psi_k = q_k + g_k^j q_j = 0$.♀ To see this, we note that: $VF + v^k g_k^j X_j F = 0$ and $Vf = (X_k f)v^k$ imply that $\{X_k f + g_k^j X_j f\}(x^k - c^k) = 0$. However, the functions $x^k - c^k$

---

♀ Note that $\Phi$ and F do not depend on v.

are independent and therefore, we have $X_k f + g_k^j X_j f = \Psi_k(\varphi_f) = 0$. Finally, on $N \cap U \times \{1\}$, we have $\varphi_F = \varphi_f = \varphi_h$ which completes the proof of the theorem.

Appendix to Section E

Let $\Phi_k = g_k^j(y^1, \ldots, y^n) q_j - g_k(y^1, \ldots, y^n) = 0$, $k = 1, \ldots, m$, be a linear system on F(M) and let U be a coordinate region in M with $x^1, \ldots, x^n$ as coordinate functions. If N is an n - m dimensional submanifold of M such that $N \cap U$ is given by the equations $x^k = c^k$, and if h is a smooth function on M such that $h = h(x^{m+1}, \ldots, x^n)$ on $N \cap U$, then we shall show that there is a linear homogeneous system $\Psi_k = 0$ on $M \times R$ such that solutions of the local initial value problem for the system $\Psi_k = 0$ will determine solutions of the system $\Phi_k = 0$ which "agree with" h on $N \cap U$.

We let v be the R-coordinate on $M \times R$ and we let r be the coordinate which corresponds to dv. Then the system $\Psi_k = g_k^j q_j + g_k r = 0$ is linear homogeneous and the function $v - h(x^{m+1}, \ldots, x^n)$ is smooth on $N \cap U \times R$. If g is a solution of the initial value problem for the system $\Psi_k = 0$ with $\varphi_g = \varphi_{v-h}$, then $D_v g \neq 0$ (since $D_v g = 1$ on the initial manifold) and consequently, $g(x^1, \ldots, x^n, v) = 0$ determines v as a smooth function f of $x^1, \ldots, x^n$. It follows that $g_k^j D_j f - g_k = g_k^j(-D_j g/D_v g) - g_k = -(g_k^j D_j g + g_k D_v g)/D_v g = 0$ and hence that f is a solution of the system $\Phi_k = 0$. Furthermore, on $N \cap U$, $f(x^1, \ldots, x^n) - h(x^{m+1}, \ldots, x^n) = 0$ so that $\varphi_f = \varphi_h$. On the other hand, if f is a solution of the system $\Phi_k = 0$ with $\varphi_f = \varphi_h$ on $N \cap U$, then $v - f = g$ is a solution of the system $\Psi_k = 0$ with $\varphi_g = \varphi_{v-h}$ on $N \cap U \times R$. Thus, we have shown that it is sufficient to solve the local initial value problem for linear homogeneous systems.

F. General Systems

We shall develop the theory of general systems of first order partial differential

equations in one dependent variable in an analogous manner to that used for linear systems. We let $\Phi_k = \Phi_k(z, q_1, \ldots, q_n, y^1, \ldots, y^n) = 0$, $k=1, \ldots, m$, (with $m \leq n$) be such a system and we suppose, in view of the results of Section C, that $Z\,\Phi_k = 0$ for all k. We let V be the variety in F(M) which is determined by the system $\Phi_k = 0$.

Definition 13: A system $\Phi_k = 0$ is algebraically independent if V is not the set of common zeros of any proper subset of $\{\Phi_k\}$. ♀ If U is a coordinate region on M, if $\| Y_j \Phi_k, Q^i \Phi_k \|$ has rank m on $\pi^{-1}(U)$ and if the nonvanishing minor of this matrix is the same throughout $\pi^{-1}(U)$, then the system $\Phi_k = 0$ is functionally independent on $\pi^{-1}(U)$.

We observe that we may assume that the $\Phi_k$ are algebraically independent. Furthermore, if the $\Phi_k$ are functionally independent on $\pi^{-1}(U)$, then they must also be algebraically independent on $\pi^{-1}(U)$.

Definition 14: A system $\Phi_k = 0$ is complete if it is functionally independent and if $(\Phi_i, \Phi_j)$ is algebraically dependent on $\{\Phi_k\}$. A system $\Phi_k = 0$ is involutory if it is functionally independent and if $(\Phi_k, \Phi_j) \equiv 0$ for all i, j.

We note that not every system can be completed, i. e., the notion of the completion of a system is particular to linear homogeneous systems. Furthermore, we observe that if $\Phi_k(\varphi_f) = 0$ for all k, then $(\Phi_i, \Phi_j)(\varphi_f) = 0$. It follows that we can only hope to find solutions for systems which are complete (or can be completed). ♀♀

Let U be a coordinate region in M with $x^1, \ldots, x^n$ as coordinate functions and

---

♀ More precisely, if $\Phi_k$ is a minimal basis of the ideal of V in the ring of smooth functions on V.

♀♀ in the sense that the chain of varieties $V_0 = \{q : \Phi_k(q) = 0\} \supset V_1 = \{q : \Phi_k(q) = 0, (\Phi_i, \Phi_j)(q) = 0\} \supset \ldots$ is finite and does not terminate with the empty variety.

suppose that det $\| Q^j \Phi_k \|$ , $j,k=1,\ldots,m$ does not vanish on $\pi^{-1}(U)$. Then the system $\Phi_k = 0$ is functionally dependent on the system $\Psi_k = q_k - \Psi_k(q_{m+1},\ldots,q_n,y^1,\ldots,y^n) = 0$ and we shall suppose that $\Phi_k = 0$ is, in fact, algebraically dependent on the system $\Psi_k = 0$.

<u>Proposition 12:</u> The system of equations $\Psi_k = 0$ is complete if and only if it is involutory.

<u>Proof:</u> It is clear that the involutoriness of the system implies the completeness of the system.

On the other hand, we observe that:

$$
\begin{aligned}
(\Psi_i, \Psi_j) &= Y_\nu (q_i - \psi_i) Q^\nu (q_j - \psi_j) - Y_\nu (q_j - \psi_j) Q^\nu (q_i - \psi_i) \\
&= - Y_j \psi_i + Y_\nu \psi_i Q^\nu \psi_k + Y_i \psi_j - Y_\nu \psi_j Q^\nu \psi_k \\
&= H(q_{m+1},\ldots,q_n,y^1,\ldots,y^n)
\end{aligned}
$$

where H is some smooth function. However, if the system is complete, we have $(\Psi_i, \Psi_j) \equiv \varphi^k_{ij} \Psi_k = \varphi^k_{ij} q_k - \varphi^k_{ij} \Psi_k$, where the $\varphi^k_{ij}$ are smooth functions on F(M), and consequently, we have $H(q_{m+1},\ldots,q_n,y^1,\ldots,y^n) + \varphi^k_{ij} \Psi_k = \varphi^k_{ij} q_k$. But the $q_\alpha$ are functionally independent. It follows that $(\Psi_i, \Psi_j) \equiv 0$.

<u>Definition 15:</u> A general system of the form $\Psi_k = q_k - \psi_k(q_{m+1},\ldots,q_n,y^1,\ldots,y^n)$, $k=1,\ldots,m$, which is complete (a fortiori involutory) is called <u>canonical</u>.

Let U be a coordinate region in M with $x^1,\ldots,x^n$ as coordinate functions and let N be an n - m dimensional submanifold of M such that $N \cap U$ is given by the equations $x^k = c^k$, $k=1,\ldots,m$. Let h be a smooth function on M whose domain contains U, and suppose that $h = h(x^{m+1},\ldots,x^n)$ on $N \cap U$. If $\Psi_k$, $k=1,\ldots,m$ is a family of smooth

functions on F(M) whose domains contain $\pi^{-1}(U)$, and if the system $\Psi_k = 0$ is canonical, then we wish to examine the question of the existence of an exact section $\varphi$ of F(M) over some $U' \subset U$ such that $\Psi_k \circ \varphi = 0$ for $k = 1, \ldots, m$ and $\varphi = \varphi_h$ on $N \cap U$. Our next theorem provides an answer to this question.

Theorem 4: If $\Psi_k = 0$ is a canonical system, then there is an open $U' \subset U$ and a smooth function f on M such that f is a solution of the local initial value problem for the system $\Psi_k = 0$ on U'.

Proof: The proof is similar to that of Theorem 3. We let I denote the interval $[0, 1]$ and we consider the mapping $\sigma$ of $M \times I$ into M given by: $\sigma((v^k(q), x^{m+j}(q), v(q))) = (c^k + v(q)v^k(q), x^{m+j}(q))$. Then $\sigma$ is a smooth mapping. We shall consider the equation $\Phi = r - v^k \Psi_k = 0$ on $M \times I$, where r is the coordinate which corresponds to dv on I, with the initial condition $H(v^k, x^{m+j}, 0) = h(x^{m+1}, \ldots, x^n)$.

We observe that the conditions of Theorem 1 hold for the equation $\Phi = 0$ and that the solution F of this equation is defined on an open set of the form $U' \times [0, a)$. However, we note that $\Phi$ is independent of v and we assert that the restriction of F to $M \times \{a'\}$, $a' < a$, is a solution of the system $\Psi_k = q_k - \Psi_k = 0$. To see this, we note that: $VF - v^k \Psi_k(\varphi_F) = 0$ and $Vf = (X_k F)v^k$ imply that $\{X_k f - \Psi_k(\varphi_f)\}(x^k - c^k) = 0$. However, the functions $x^k - c^k$ are independent and therefore, we have $X_k f - \Psi_k(\varphi_f) = 0$. Finally, on $N \cap U \times \{a'\}$, we have $\varphi_F = \varphi_f = \varphi_h$ which completes the proof of the theorem.

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## Distribution List

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